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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series is designed to make material for the study of topics of special interest to students readily accessible in classroom quantity. Topics covered include points, lines, space, planes, segments, separations, angles, one-to-one correspondence, and simple closed curves. (MP)

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SCHOOL
MATHEMATICS
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SP-8

SUPPLEMENTARY and ENRICHMENT SERIES

NON-METRIC GEOMETRY

Edited by Ronald J. Clark

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SMSG
SE 008 665

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PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

Prepared under the supervision of the Panel on Supplementary Publications of the School Mathematics Study Group:

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NON-METRIC GEOMETRY

Introduction

Much of mathematics is concerned with numbers and their properties, but numbers are not the only things in mathematics that interest people. Living as we do in the "Space Age", we hear much about points, lines, planes, and space. The study of concepts like these is called geometry.

For over 4,000 years men, in trying to understand better the world in which they lived, have studied geometry. The geometry we study today, called "Euclidean" geometry because a famous Greek mathematician named Euclid organized (about 300 B. C.) all that was then known into several books, is very much like what the Greeks studied although some of our words and ways of looking at things are different.

Although geometry may involve the idea of distance and measurement, it is not necessary to do this. A great deal of geometry, usually called non-metric geometry, does not need concepts of numbers and measurement. In this pamphlet, we shall study such geometry and discover some of the many interesting ideas about points, lines and space.

Today when we are sending rockets toward the moon and satellites into orbit, the study of geometrical ideas like these is more important than ever.

1. Points, Lines, and Space

Points. The idea of a point in geometry is suggested by the tip of a pencil or a dot on the chalkboard. A geometric point is thought of as being so small that it has no size. In geometry we do not give a definition for the term "point." What we do instead is to describe many properties of points. In this way we come to understand what mathematicians mean by the term "point."

Space. We think of space as being a set of points. There is an unlimited quantity of points in space. In a way, we think of the points of space as being described or determined by position--whether they are in this room, in this world, or in the universe.

Lines. For us, a line is a set of points in space, not any set of points but a particular type of set of points. The term "line" means "straight line." All lines in our geometry are understood to be straight. A line is suggested by the edge of a ruler. It is suggested by the intersection of a side wall and the front wall of your classroom.

A geometric line extends without limit in each of two directions. It does not stop at a point. The intersection of a side wall and the front wall of your classroom stops at the floor and the ceiling. The line suggested by that intersection extends both up and down, indefinitely far.

You have probably heard people say, "as the crow flies." A crow usually flies directly from one landing point to another. The expression therefore means "in a straight line." Crows do not flit about as bats do. The path of flight of a crow, then, suggests a geometric line. We should understand that the "line" does not start or stop at the crow's resting places. It extends endlessly in both directions.

Think about two students holding a string stretched between them. In any position where it is straight, the taut string marks a portion of a line. It is the line through one student's fingers and also through the other's. The line itself goes beyond where they hold the string. The string is not the line or any part of a line. It just represents the line we know to be there.

With the students' fingers in the same positions, is there more than one possible position for the taut string? You probably think, "Of course not," and you are right. You now have discovered a basic property of space.

Property 1: Through any two different points in space there is exactly one line.

As you can see, there are an unlimited number of lines in space!

By using lines we can get a good idea of what space is like. Consider a point at a top corner of your teacher's desk. Now consider the set of all points suggested by the walls, the floor, and the ceiling of your classroom. Then for each point of this set there is a line through it and the selected point on your teacher's desk. Each line is a set of points. Space is made up of all the points on all such lines. Remember, these lines extend outside the room.

Just as we do not precisely define "point" and "line," we do not precisely define "space." We study its properties and in this way understand it. An idea of space was known to the ancient Greeks who worked with the concept. We too can only really understand ideas like those of space and point after a great deal of study. We can't expect to learn everything about them in a few days, a week, or even this year.

Class Discussion Problems

1. Consider one of the lines that pass through the pencil sharpener and the knob of the entrance door to your classroom. What objects in the room are "pierced" by this line? What objects outside the room are also "pierced" by this line?

2. A mathematical poet might say, "Space is like the bristling, spiny porcupine." In what ways is this description like that in the discussion following the statement of Property 1?

Exercises 1

1. For party decorations, crepe-paper ribbons were draped between two points on the gymnasium walls. Does this show, contrary to Property 1, that there may be several geometric lines through two points?

2. When a surveyor marks the boundaries of a piece of land, he places stone "monuments" (small stone blocks) at the corners. A small hole or nail in the top of each monument represents a point. He knows that, if the monuments are not moved out of position, the original boundaries may be determined at any later time. How can he be sure of this?

3. A harp or violin player must learn exactly where to place his fingers on the strings of his instrument to produce the sounds he wishes to hear. Sometimes a string will break and be replaced by a new one. How does he know, without looking, where to place his fingers so that they will rest on the new string?

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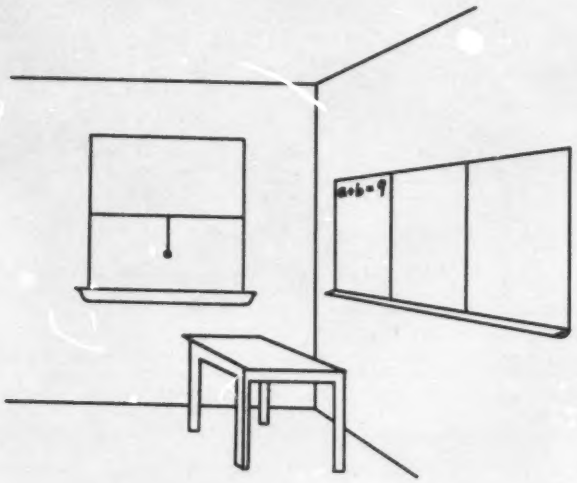
2. Planes

Any flat surface such as the wall of a room, the floor, the top of a desk, a piece of plywood, or a door in any position suggests the idea of a plane in mathematics. Like a line in mathematics a plane is thought of as being unlimited in extent. If you begin at a starting point on a plane and follow paths on the plane in all possible directions, these paths will remain in the plane but have no endings. A plane, then, would have no boundaries!

We think of a plane as containing many points and many lines. As you look at the wall, you can think of many points on it, and you can also think of the lines containing these points. The edge where the side wall of a room meets the ceiling suggests a line in either the plane represented by the

ceiling or the plane represented by the side wall. The edges of the chalk tray represent lines in the classroom. At least one of these is in the plane represented by the chalkboard. Any number of points and lines could be marked on the chalkboard to represent points and lines in geometry.



Mathematicians think of a plane as a set of points in space. It is not just any set of points, but a particular kind of set. We have already seen that a line is a set of points in space, a particular kind of set and a different kind from that of a plane. A plane is a mathematician's way of thinking about the "ideal" of a flat surface.

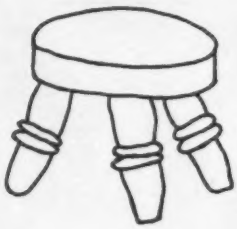
If two points are marked on the chalkboard, exactly one line can be drawn through these points, Is this just what Property 1 says? This line is on the chalkboard. The plane represented by the chalkboard contains the set of points represented by the line which you have drawn.

Think of two points marked on a piece of plywood. Part of the line through these points can be drawn on the plywood (recall that "line" means "straight line"). Must the line through these two points be on the plane of the plywood? We can now conclude:

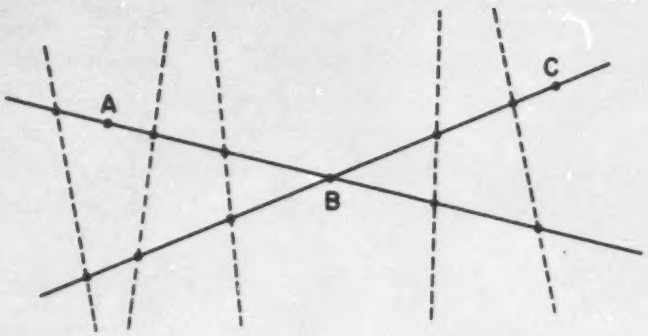
Property 2: If a line contains two different points of a plane, it lies in the plane.

Notice a pair of corners of the ceiling in the front of your room. In how many planes is this pair of points contained? If one thinks of the ends of the binding of his tablet as a pair of points, he can see that the planes represented by the pages of the tablet contain these points.

The question might be asked, "How many planes contain a specified pair of points?" The tablet with its sheets spread apart suggests that there are many planes through a specified pair of points. The front wall and the ceiling represent two planes through the point represented by the two upper front corners of the room. Show by means of hand motions the positions of other planes through these two points. A door in several positions represents many planes passing through a pair of points--its hinges. We can say, "Many planes contain a pair of points."



Suppose next that we have three points not all on the same line. Three corners of the top of your teacher's desk is an example of this. The bottoms of the legs of a three-legged stool is another example. Such a stool will stand solidly against the floor, while a four-legged chair does not always do this unless it is very well constructed. Spread out the thumb and first two fingers of your right hand as in the figure above. Hold them stiffly and think about their tips as being points. Now take a book or other flat surface and attempt to place it so that it lies on the tips of your thumb and two fingers (on the three points). Can you hold the book against the tips of your thumb and two fingers? If you bend your wrist and change the position of your thumb and fingers, will the book be in a new position? With your hand in any one position, is there more than one way in which a flat surface can be held against the tips of your fingers? It seems clear that there is only one position for the flat surface.



Property 3: Any three points not on the same line are in only one plane.

Do you see why this property suggests that if the legs of a chair are not exactly the same length that you may be able to rest the chair on three legs, but not on four?

In the figure above, there are three points, A, B, and C in a plane. The line through points A and B and the line through points B and C are drawn. The dotted lines are drawn so that they contain two points of the plane of A, B, and C. Each dotted line contains a point of the line through A and B, and a point of the line through B and C. The dotted lines are contained in the plane. Which property says this? The sets of points represented by the dotted lines are contained in the plane. The plane which contains A, B, and C can now be described. It is the set of all points which are on lines containing two points of the figure consisting of the lines through A and B and through B and C.

Class Discussion Problems

1. A plane contains three points suggested by the two front feet of the teacher's desk and the pencil sharpener. Through what objects in the room does this plane pass? What objects outside of the room?

2. Place a ruler, edge up, upon your desk. Keep one edge of a surface, such as a piece of cardboard, in moving contact with the desk top but at the same time keep the surface in contact with the upper edge of the ruler. Make the surface slope gradually, then steeply. In how many different sloping positions may the "surface" be placed? On your desk, near but not in line with the ruler, place an eraser, tack, marble, or some other small object that would suggest a point. Place the surface on this object and the upper edge of the ruler. In how many different sloping positions may the surface be placed this time? How do these two situations show what is meant by Property 2 and Property 3?

Exercises 2

1. In a certain arrangement of three different points in space, the points can be found together in each and every one of many different planes. How are the points arranged?

2. In another arrangement, three different points can be found together in only one plane. How are the points arranged?

3. Photographers, surveyors, and artists generally use tripods to support their equipment. Why is a three-legged device a better choice for this purpose than one with four legs? How does Property 3 apply here?

4. How many different lines may contain:
   (a) One certain point?
   (b) A certain pair of points?

5. How many different planes may contain:
   (a) One certain point?
   (b) A certain pair of points?
   (c) A certain set of three points?

6. Why is the word "plane" used in the following names: airplane, aquaplane? Find out what the dictionary states about plane sailing, plane table, planography.

* 7. How many lines can be drawn through four points, a pair of them at a time, if the points lie:
   (a) In the same plane?
   (b) Not in the same plane?

*8. Explain the following: If two different lines intersect, one and only one plane contains both lines.

---

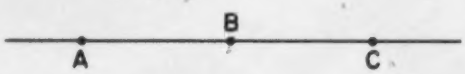
3. Names and Symbols

It is customary to assign a letter to a point and thereafter to say "point A" or "point B" according to the letter assigned. Short-cut arrangements like this are useful, but we should be certain that their meanings are clearly understood.

A dot represents a point. We shall tell which point we have in mind by placing a letter (usually a capital) as near it as possible. In the figure below, which point is nearest the left margin? Which point is nearest the right margin?

A line may be represented in two ways, like this or simply like this . In the first drawing, what do the arrowheads suggest? Does a line extend indefinitely in two directions? The drawing, then, suggests all the points of the line, not just those that can be indicated on the page.

If we wish to call attention to several points on a line, we shall do it this way:

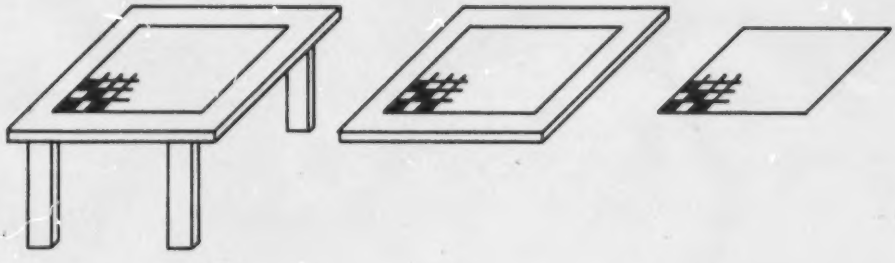


and the line may be called "line AB." A symbol for this same line is $\overleftrightarrow{AB}$. Other names for the above line are "line AC" or "line BC." The corresponding symbols would be $\overleftrightarrow{AC}$ or $\overleftrightarrow{BC}$.

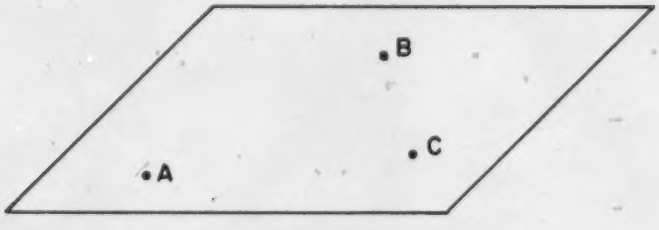
Notice how frequently the word "represent" appears in these explanations. A point is merely "represented" by a dot because as long as the dot mark is visible, it has size. But a point, in geometry, has no size. Also lines drawn with chalk are rather wide, wavy, and generally irregular. Are actual geometric lines like this? Recall that "line" for us means "straight line."

A drawing of a line by a very sharp pencil on very smooth paper is more like our idea of a line, but its imperfections will appear under a magnifying glass. Thus, by a dot we merely indicate the position of a point. A drawing of a line merely represents the line. The drawing is not the actual line. It is not wrong to draw a line free-hand (without a ruler or straight edge) but we should be reasonably careful in doing so.

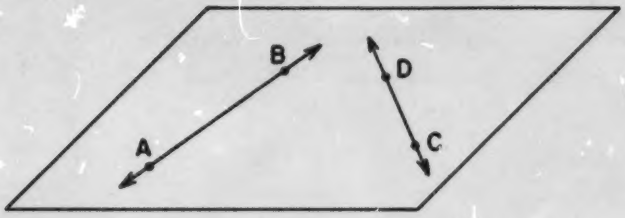
Just as we need to represent point and line, we find it necessary to "represent" a plane. Figure (a) is a picture of a checkboard resting on top of a card table; Figure (b) the same with the legs removed and Figure (c) the checkerboard alone.



A table top suggests a portion of a plane. In this case the checkerboard suggests a smaller portion and the individual sections of the checkerboard still smaller portions of the plane. If we want to make a drawing on a piece of paper, we use a diamond-shaped figure as below. Points of a plane are indicated in the same way as points of a line:

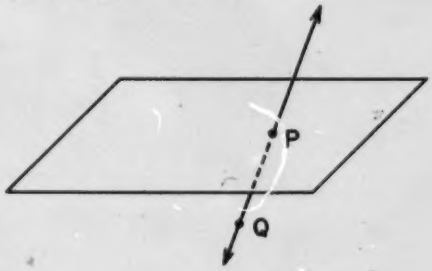


In the figure above, do points A and C seem closer to you than point B? If not, imagine point B as an opponent's checker at the far edge of the checkerboard. Then A and C would be checkers belonging to you.



In the above figure, A, B, C, and D are considered to be points in the plane of the checkerboard. A line is drawn through point A and point B. Another line is drawn through point C and point D. According to Property 2, what can be said about $\overleftrightarrow{AB}$? Property 2 states, "If a line contains two points of a plane, it lies in the plane." What can be said about $\overleftrightarrow{DC}$?

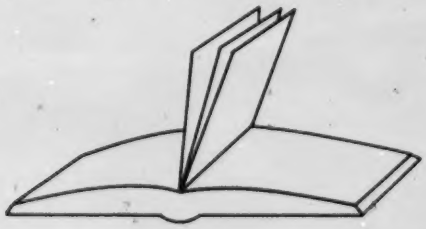
It is possible that a line might "pierce" or "puncture" a plane. A picture of this situation may appear thus:



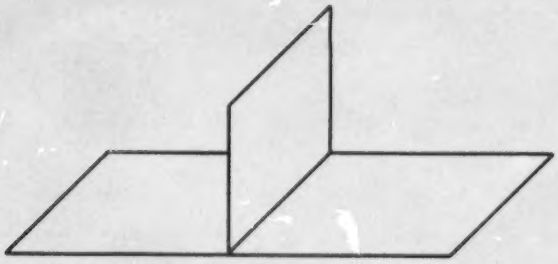
The dotted portion of $\overleftrightarrow{PQ}$ would be hidden from view if the part of the plane represented were the upper surface of some object such as the card table.

Once again, the drawing only "represents" the situation.

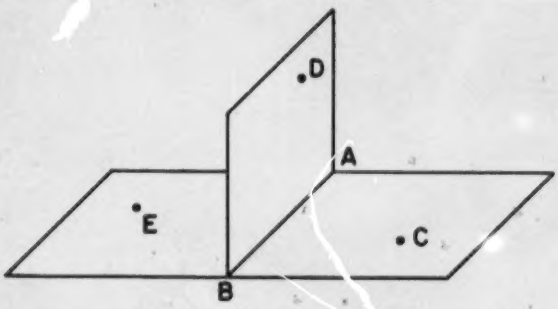
This is a drawing of an open book:



If all pages of the book except one of the upright set were removed, then the drawing would change to this:



This now suggests the intersection of two planes, one containing the front and back covers and the other containing the single page. Since there is more than one plane in the figure, it now becomes useful to label them. Let us assign letters to some of the points of the diagram.



How could we use Property 3 to identify the planes of the figure above? Remember that Property 3 states, "Any three points not all on the same line are in exactly one plane." Would the letters ABD suggest the upright page or the book cover? Note that point A, point B, and point D are "not all on the same line." Many people would agree that the upright page is suggested by these letters. The page in question suggests a plane and may now be designated as "plane ABD." Following this same arrangement, the book cover may be called, "plane ABC," but it seems that "plane ABE" would be another way of indicating the book cover! To show that apparently two names for the same plane may be given, it might be possible to write plane ABC = plane ABE if we are certain of what the equal sign means as used here.

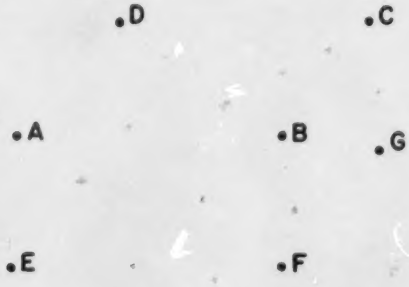
The notion of "set" will be helpful in explaining what is meant by "equal" when applied this way. Let's see what facts can be ascertained about the situation described by the figure:

1. Plane ABC is a set of points extending beyond the book cover.
2. Plane ABE is a set of points extending beyond the book cover.
3. Points A, B, C, E, and others not indicated are in plane ABC and are also in plane ABE.

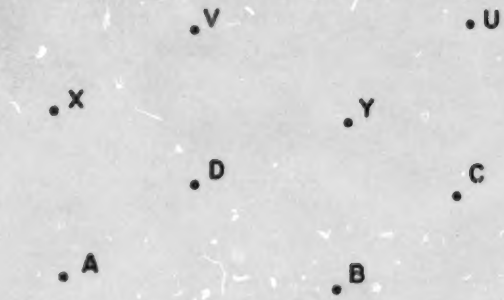
In other words, all elements of plane ABC, (a set of points) and elements of plane ABE (a set of points) seem to be contained in both sets (planes). We shall say, "Two sets are <u>equal</u> if and only if they contain the same elements." According to this, plane ABC = plane ABE. In other words, we say set M is equal to set N if M and N are two names for the same set.

Exercises 3

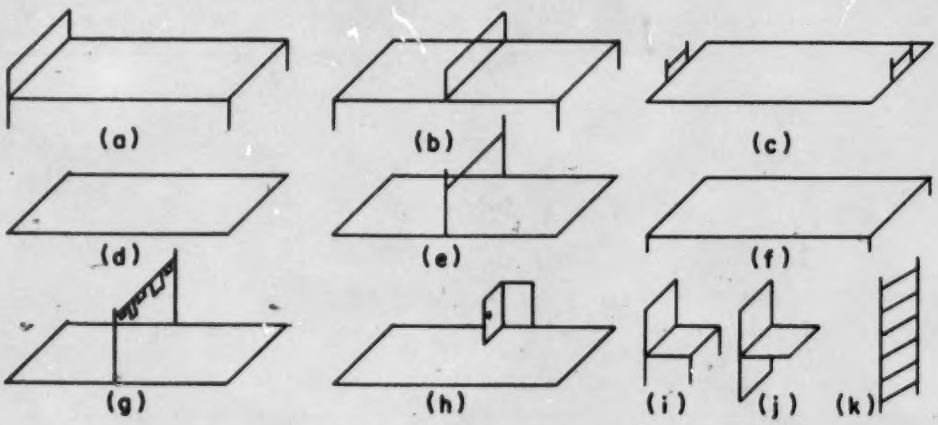
1. As one usually holds a page, which indicated point of the figure above is to the left of CD? To the right? Which indicated point is nearest the top of the page? The bottom?



2. Transfer the points in the above figure to a piece of paper by tracing. From now on we shall refer to the copy you have made. With a portion of a line, join A to B, then B to C, C to D, D to A in that order. Now join A to E, B to F, C to G. What familiar piece of furniture might this sketch represent?



3. Make a tracing of the above figure. Join A to B, B to C, C to D, D to A, A to X, B to Y, D to V, and C to U. What has happened to the table?



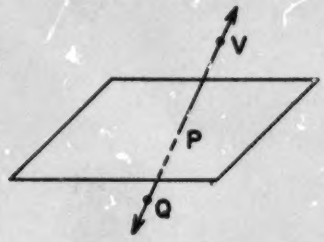
4. Each of the above sketches represents one of the familiar objects listed below. Match the sketches with the names.

Cot	Football field	Line of laundry
Ping Pong table	High jump	Ladder
Carpet	Coffee table	Open door
Chair		Shelf

5. Make sketches representing each of the following:

Desk	Pyramid with square base
Cube	Cereal box

6. In the figure below, is point V a point of $\overleftrightarrow{PQ}$? Is point Q an element of the plane? Is V? How many points of $\overleftrightarrow{PQ}$ are elements of the plane?



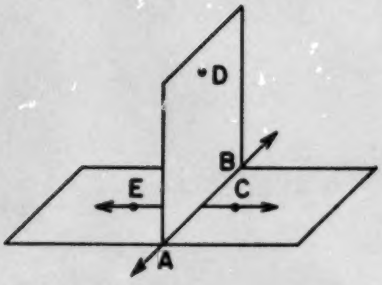


Figure (a)

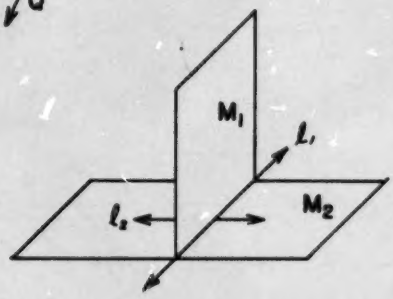


Figure (b)

7. Figure (b) is a copy of Figure (a), except for labeling. Three boys named "Tom," who are in the same class might be called T_1, T_2, and T_3 to avoid confusing one with the other. Similarly, three different lines may be denoted as ℓ_1, ℓ_2, and ℓ_3. The small numbers are not exponents. They are called "subscripts." Plane ABD in Figure (a) corresponds to M_1 in Figure (b). $\overleftrightarrow{AB}$ in Figure (a) corresponds to ℓ_1 in Figure (b).

In the left-hand column are listed parts of Figure (a). Match these with parts of Figure (b) listed in the right-hand column:

Parts of Figure (a)	Parts of Figure (b) *
1. $\overleftrightarrow{EC}$	a. ℓ_1
2. Plane ABC	b. ℓ_2
3. Plane ABD	c. M_1
4. Plane EBA	d. M_2
5. $\overleftrightarrow{AB}$	
6. The intersection of plane ABC and plane ABD	

* Does the second column suggest an advantage of the subscript way of labeling?

8. In the figure at the right,

(a) Does ℓ_1 pierce M_1?

(b) Also M_2?

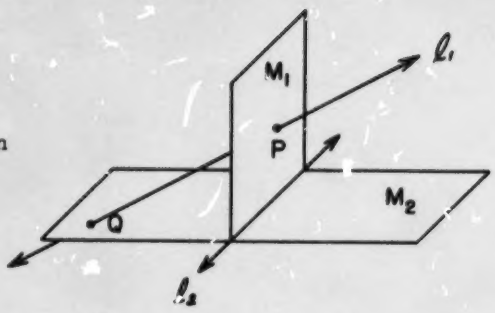
(c) Is ℓ_1 the only line through P and Q?

(d) What is the intersection of M_1 and M_2?

(e) Is ℓ_1 in M_2?

(f) Would ℓ_1 meet ℓ_2?

(g) Are ℓ_1 and ℓ_2 in the same plane?



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4. Intersection of Sets

We now shall introduce some useful and important ideas about sets.

Let set A = {1,2,3,4,5,6,7,8,9}

Let set B = {2,4,6,8,10,12,14,16}

Let set C be the set of those elements which are in set A and are also in set B. We can write set C = {2,4,6,8}. We call C the intersection of set A and set B.

Let set R be the set of pupils with red hair.

Let set S be the set of pupils who can swim.

It might happen that an element of set R (a pupil with red hair) might be an element of set S (a pupil who can swim). In fact, there may be no such common elements or there may be several. In any case, the set of red-headed swimmers is the intersection of set S and set R.

A set with no elements in it is called the "empty set." Thus, if there are no red-headed swimmers, then the intersection of set S and set R is the empty set.

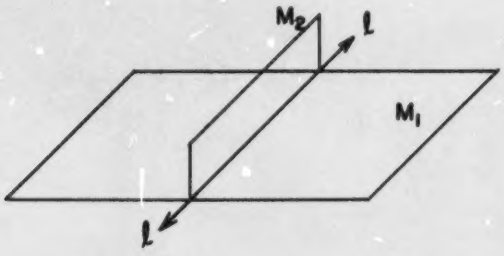
Let set H be the set of pupils in your classroom and let set K be the set of people under two years of age. Then the intersection of H and K is the empty set. There are no pupils in your classroom under two years of age!

We use the symbol $\cap$ to mean "intersection," that is, $E \cap F$ means "the intersection of set E and set F." Thus, referring to the sets mentioned previously, we write:

$A \cap B$ is $\{2,4,6,8\}$

$R \cap S$ is the set of red-headed swimmers

$H \cap K$ is the empty set.



The figure above suggests two planes M_1 and M_2. (M_1 is like a ping-pong table. M_2 is like the net.) The line ℓ seems to be in M_1 and also in M_2. Every point in M_1 which is also in M_2 seems to be on the line ℓ. Thus the following statement seems to be true: $M_1 \cap M_2 = \ell$.

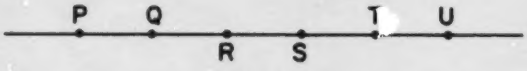
Some confusion occasionally arises around the use of the words "intersection" and "intersect". We can always talk about the intersection of two sets, even if the two sets have no elements in common, in which case we say the intersection is empty, or is the empty set. The word "intersect" is most often used when referring to the representations of sets. Thus we say that two lines do not intersect. We can say, however, in this case that the intersection of the two sets is empty.

At times, when we are hurrying or are discussing a problem without being particularly careful, difficulties can arise. Then it is necessary to be precise and identify whether we are referring to diagrams or the sets. The use of the word intersection is usually more precise because we can talk about empty sets quite rationally. The use of the word intersect varies with authors; we shall try to avoid it except when referring to physical interpretations.

Exercises 4

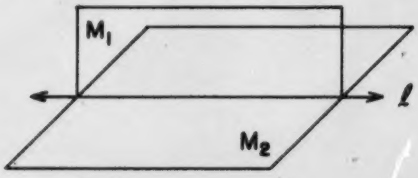
1. Write the set whose members are:
   (a) The whole numbers greater than 17 and less than 23.
   (b) The cities over 100,000 in population in your state.
   (c) The members of the class less than 4 years old.

2. Write three elements of each of the following sets:
   (a) The odd whole numbers
   (b) The whole numbers divisible by 5
   (c) The set of points on the line below, some of which are labeled in the figure:



3. Give the elements of the intersections of the following pairs of sets:
   (a) The whole numbers 2 through 12 and the whole numbers 9 through 20
   (b) The members of the class and the girls with blond hair
   (c) The set of points on line k_1 and the set of points on line k_2

   (d) The set of points on plane M_1 and the set of points on plane M_2



4. Let $S = \{4,8,10,12,15,20,23\}$
   $T = \{2,7,10,13,15,21,23\}$
   Find $S \cap T$.

5. Think of the top, bottom, and sides of a chalk box as sets of points.
   (a) What is the intersection of two sides that meet?
   (b) What is the intersection of the top and bottom?

6. Let S be the set of New England States, T be the set of states whose names begin with the word "New," and V be the set of states which border Mexico.
   (a) List the states in the three sets, S, T, and V using the { } notation.
   (b) What is $S \cap T$?
   (c) What is $S \cap V$?
   (d) What is $V \cap T$?

*7. The set of whole numbers which are multiples of 3 is closed under addition.
   (a) Is the set of whole numbers which are multiples of 5 closed under addition?
   (b) Is the intersection of the sets described in this exercise closed under addition? Why?

*8. Explain why "intersection" has the closure property and is both commutative and associative. In other words, if X, Y, and Z are sets, explain why:
   (a) $X \cap Y$ is a set.
   (b) $X \cap Y = Y \cap X$.
   (c) $(X \cap Y) \cap Z = X \cap (Y \cap Z)$.

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5. Intersections of Lines and Planes

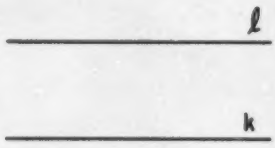
Two Lines. Look at a chalk box and think of the edges as representing lines. Some of these lines have points in common and some do not. On the top of the box are some lines (edges) which have points in common and some which do not. If we think of the lines contained in the top and the bottom of the box, we see pairs which do not have points in common but have the same direction, and pairs which do not have points in common and do not have the same direction. Is this situation also true of the edges in your classroom? (An edge is the line of intersection of two walls, a wall and the ceiling, or a wall and the floor.)

Can you hold your two arms in a position so that they represent intersecting lines? Can you hold them in a position so that the lines they represent do not intersect but have the same direction? Can you hold them in a position so that the lines they represent do not intersect and do not have the same direction? Are there any other possibilities as far as intersections of pairs of lines are concerned?

The possible intersections of two different lines may be described in three cases. Figures are drawn to represent the first two cases but not the third. As you read the third case, can you think of the reason it is difficult to represent it in a figure? Look at the figure for Problem 8 of Exercises 3. Could this be used for case three?

Case 1. ℓ and k intersect, or $\ell \cap k$ is not the empty set. ℓ and k cannot contain the same two points. Why?

Case 2. ℓ and k do not intersect and are in the same plane. $\ell \cap k$ is the empty set, and ℓ and k are in the same plane. ℓ and k are said to be parallel.



Case 3. ℓ and k do not intersect and are not in the same plane. $\ell \cap k$ is the empty set, and ℓ and k are not in the same plane. We say that ℓ and k are skew lines.

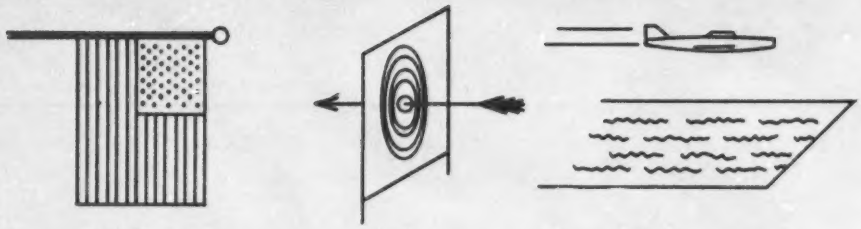
Exercise 8, section 2, asks you to explain why two lines lie in the same

plane if they intersect. In the figure above are shown two lines which intersect in point A. B is a point on one of the lines and C a point on the other. By Property 3, there is exactly one plane which contains A, B, and C.

By Property 2, AB is in this plane.

By Property 2, AC is in this plane.

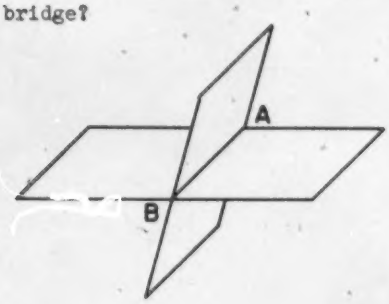
There is exactly one plane which contains the two lines

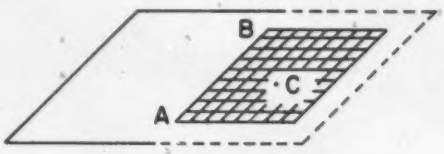


(a) (b) (c)

A Line and a Plane. There is a way to arrange a line and a plane so that their intersection contains only one point. Does one of the drawings above suggest this way? There is another way to arrange a line and plane so that their intersection contains many points! Which drawing suggests this way? There is still another way to arrange a line and a plane so that their intersection contains no points at all! If the first two drawings were chosen correctly you won't have difficulty in choosing the correct drawing this time. If we refer to each of the above arrangements as a "case," then these three "cases" might be suggested also by the sides and edges of a chalk box, a shoe box, or the walls and edges of the room.

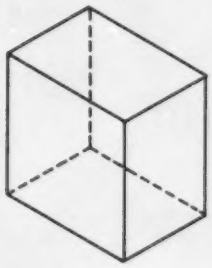
Two Planes. Next, let us think of two different planes in space. Suppose their intersection is not empty. Does the intersection contain more than one point? Notice that the planes of the front wall and of a side wall of a room intersect in more than one point. If you have two sheets of paper and you hold a sheet of paper in each hand, it might seem that the pieces or sheets of paper could be held so that they have only one point of intersection. But if we consider the planes of the sheets of paper and not just the sheets themselves, we see that if they have one point in common, it follows that their intersection will necessarily contain other points. Can you hold the two sheets so that they are flat and still represent planes that would intersect in only two points? Keeping the sheets as flat as possible, can you hold them so that they intersect in a curved line like the arch of a bridge?





Let A and B be two points, each of which lies in two intersecting planes as in Figure (d). From Property 2, the line $\overleftrightarrow{AB}$ must lie in each of the planes. Hence the intersection of the two planes contains a line. But if, as in Figure (e), the intersection contains a point C not on the line $\overleftrightarrow{AB}$, then the two planes would be the same plane. By Property 3 there would be exactly one plane containing A, B, C. We now state:

Property 4: If the intersection of two different planes is not empty, then the intersection is a line.



If the intersection of two planes is the empty set, then the planes are said to be parallel. Several examples of pairs of parallel planes are represented by certain walls of a room or a stack of shelves. Can you think of others?

In discussing the intersection of two different planes we have considered two cases. Let M and N denote the two planes.

Case 1. $M \cap N$ is not empty. $M \cap N$ is a line.

Case 2. $M \cap N$ is empty. M and N are parallel.

Are there any other cases? Why?

Exercises 5

1. List the three cases for the intersection of a line and a plane. Indicate whether the intersection contains 0, 1 or more than 1 point.
2. Describe two pairs of skew lines (lines that do not lie in the same plane) suggested by edges in your room.
3. On your paper, label three points A, B, and C so that $\overleftrightarrow{AB}$ is not $\overleftrightarrow{AC}$. Draw the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$. What is $\overleftrightarrow{AB} \cap \overleftrightarrow{AC}$?

4. Carefully fold a piece of paper in half. Does the fold suggest a line? Stand the folded paper up on a table (or desk) so that the fold does not touch the table. The paper makes sort of a tent. Do the table top and the folded paper suggest three planes? Is any point in all three planes? What is the intersection of all three planes? Are any two of the planes parallel?

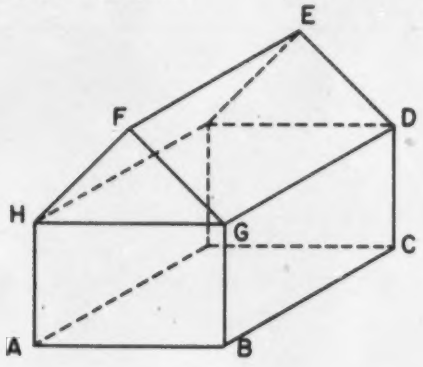
5. Stand the folded paper up on a table with one end of the fold touching the table. Are three planes suggested? Is any point in all three planes? What is the intersection of the three planes?

6. Hold the folded paper so that just the fold is on the table top. Are three planes suggested? Is any point in all three planes? What is the intersection of the three planes?

7. In each of the situations of Exercises 4, 5, and 6, consider only the line of the fold and the plane of the table top. What are the intersections of this line and this plane? You should have three (different?) answers, one for each of Exercises 4, 5, and 6.

8. Consider three different lines in a plane. How many points would there be with each point on at least two of the lines? Draw four figures showing how 0, 1, 2, or 3 might have been your answer. Why couldn't your answer have been 4 points?

9. Consider this sketch of a house.



We have labeled eight points on the figure. Think of the lines and planes suggested by the figure. Name lines by a pair of points and planes by three points. Name:

(a) A pair of parallel planes.

(b) A pair of planes whose intersection is a line.

(c) Three planes that intersect in a point.

(d) Three planes that intersect in a line.

(e) A line and a plane that do not intersect.

(f) A pair of parallel lines.

(g) A pair of skew lines.

(h) Three lines that intersect in a point.

(i) Four planes that have exactly one point in common.

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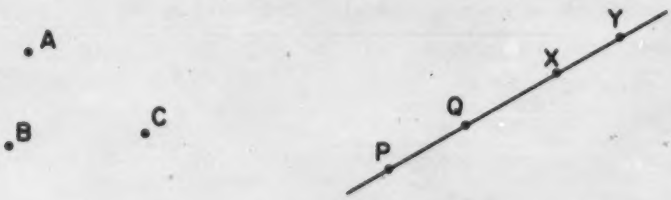
6. Segments

Consider three points A, B, and C as in the figure below. Do we say that any one of them is between the other two? No, we usually do not.



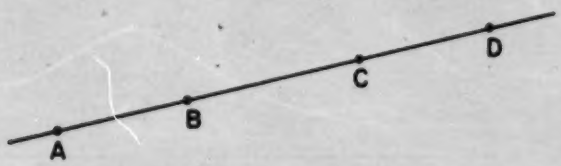
We use the word "between" only when the points in question are on the same line. Look at points P, Q, and Y above. These points represent points on a line. Is X between Q and Y? Is Q between P and Y? Is P between X and Y? If you have said yes, yes, and no in that order, you are correct. All of us have a good natural sense of what it means to say that a point is between two other points on a line.

You will observe, when we say that a point P is between points A and B, that we mean two things: First, there is a line containing A, B, and P. Second, on that line, P is between A and B.

Let us look at the figure again. Is there some point between P and Q? We have not labeled any, but we understand there are many there. In fact, for any two points, A and B, in space we understand the situation is like that for P and Q. There is a line containing A and B, and on this line, there are points between A and B.

Finally, we are able to say what we mean by a segment. Think of two different points A and B. The set of points consisting of A, B, and all points between A and B is called the line segment AB. A and B are called the endpoints. We name the line segment with endpoints A and B by $\overline{AB}$. Another name for this line segment is $\overline{BA}$.

Every line segment has exactly two endpoints. Clearly each line segment contains points other than its endpoints. Sometimes a line segment is called simply a segment. It is not wrong to do so, if we understand, always, that a segment is a part of a line.



In the figure above, we can name segments $\overline{AB}$, $\overline{CB}$, and $\overline{CD}$. Are there other segments? Yes there are segments $\overline{CA}$, $\overline{AD}$, and $\overline{BD}$. You recall that earlier we learned something about intersection of sets. What is the intersection of $\overline{AC}$ and $\overline{BD}$?

Not only may we talk about intersection of sets, but we also find it convenient to talk about the union of sets. The word "union" suggests uniting or combining two sets into a new set. The union of two sets consists of those objects which belong to at least one of the two sets. For example, in the figure above, the union of $\overline{AB}$ and $\overline{BC}$ consists of all points of $\overline{AB}$, together with all points of $\overline{BC}$, that is, the segment $\overline{AC}$.

We use the symbol $\cup$ to mean "union." That is, $X \cup Y$ means "the union of set X and set Y." Suppose that set X is the set of numbers $\{1,2,3,4\}$ and set Y is the set of numbers $\{2,4,6,8,10\}$. Do you have any idea of what $X \cup Y$ is? Yes, it is $\{1,2,3,4,6,8,10\}$. In the union of two sets, we do not think of an element which occurs in both sets as appearing twice in the union. There is one element that is a member of two different sets; the union has the one element.

Again, let us think of the set of all pupils who have red hair and the set of all pupils who can swim. We may think:

Let set R be the set of pupils with red hair.
Let set S be the set of pupils who can swim.

Then $R \cup S$ is the set of all pupils who either have red hair (whether or not they can swim) or who can swim (whether or not they have red hair). John, who is a boy with red hair, is a member of set R. Since John swims, he is a member of set S. The set $R \cup S$ has John listed only once.

Exercises 6

1. Draw a horizontal line. Label four points on it P, Q, R, and S in that order from left to right. Name two segments:
   (a) Whose intersection is a segment.
   (b) Whose intersection is a point.
   (c) Whose intersection is empty.
   (d) Whose union is <u>not</u> a segment.

2. Draw a line. Label three points of the line A, B, and C with B between A and C.
   (a) What is $\overline{AB} \cap \overline{BC}$?
   (b) What is $\overline{AC} \cap \overline{BC}$?
   (c) What is $\overline{AB} \cup \overline{BC}$?
   (d) What is $\overline{AB} \cup \overline{AC}$?

3. Draw a segment. Label its endpoints X and Y. Is there a pair of points of $\overline{XY}$ with Y between them? Is there a pair of points of $\overleftrightarrow{XY}$ with Y between them?

4. Draw two segments $\overline{AB}$ and $\overline{CD}$ for which $\overline{AB} \cap \overline{CD}$ is empty but $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is one point.

5. Draw two segments $\overline{PQ}$ and $\overline{RS}$ for which $\overline{PQ} \cap \overline{RS}$ is empty, but $\overleftrightarrow{PQ}$ is $\overleftrightarrow{RS}$.

6. Let A and B be two points. Is it true that there is exactly one segment containing A and B? Draw a figure explaining this problem and your answer.

7. Draw a vertical line ℓ. Label A and B two points to the right of ℓ. Label C a point to the left of ℓ. Is $\overline{AB} \cap \ell$ empty? Is $\overline{AC} \cap \ell$ empty?

8. In some old-fashioned geometry books the authors did not make any distinction between a line and a line segment. They called each a "straight line." With "straight line" meaning either of these things, explain why we cannot say that "through any two points there is exactly one straight line."

*9. Explain why "union" has the closure property and is both commutative and associative. In other words, if X, Y, and Z are sets, explain why:
   (a) $X \cup Y$ is a set.
   (b) $X \cup Y = Y \cup X$.
   (c) $(X \cup Y) \cup Z = X \cup (Y \cup Z)$.

*10. Show that for every set X, we have:

$$X \cup X = X$$

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7. Separations

In this section we shall consider a very important idea--the idea of separation. We shall see this idea applied in three different cases.

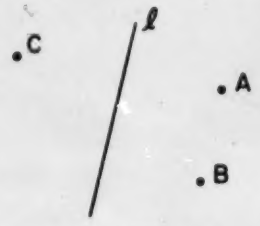
Let M be the name of the plane of the front chalkboard. This plane separates space into two sets: (1) the set of points on your side of the plane of the chalkboard, and (2) the set of points on the far side (beyond the chalkboard from you). These two sets are called half-spaces. The plane M is not in either half-space.

Let A and B be any two points of space not in the plane M of the chalkboard. Then A and B are on the same side of the plane M if the intersection of $\overline{AB}$ and M is empty, that is, if $\overline{AB} \cap M$ is empty. Also, A and B are on opposite sides of the plane M if the intersection of $\overline{AB}$ and M is not empty; in other words, there is a point of M between A and B.

Any plane M separates space into two half-spaces.

If A and B are in the same half-space, $\overline{AB} \cap M$ is empty. If A and B are in different half-spaces, $\overline{AB} \cap M$ is not empty. We call M the boundary of each of the half-spaces.

Now consider only the plane M of the front chalkboard. Do you see how the plane M itself could be separated into two half-planes? What could be the boundary of the two half-planes? Look at the next figure consisting of line, ℓ,



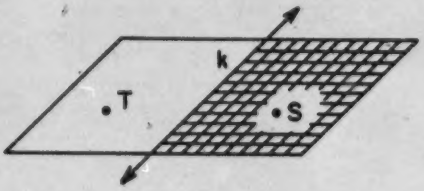
and three points, A, B, and C. Is $\overline{AB} \cap \ell$ the empty set? Is $\overline{BC} \cap \ell$ the empty set? What about $\overline{AC} \cap \ell$? We may say that:

Any line ℓ of plane M separates the plane into two half-planes.

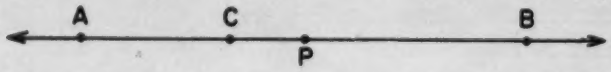
We call the two half-planes into which ℓ separates M, the sides of ℓ. We indicate the sides of ℓ by referring to the A-side of ℓ or the

C-side of ℓ. Notice that in the above figure, the B-side of ℓ = the A-side of ℓ. The line ℓ is not in either half-plane.

In the figure below the S-side of line k is shaded and the T-side of k is not shaded.



Now consider a line ℓ. How would you define a half-line? Can you say anything about segments in this definition as we did in defining half-planes and half-spaces? What would the boundary be? Is the boundary a set of points?



If point P separates the line in the figure into two half-lines, are A and B on the same half-line? Are A and C on the same half-line? Is P between B and C?

Our third case should now be clear. Can you state it?

It is important to note that these three cases are almost alike. They deal with the same idea in different situations. Thus:

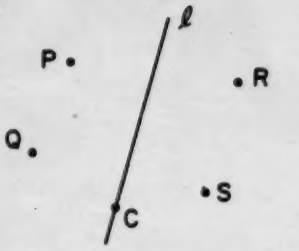
(1) Any plane separates space into two half-spaces.
(2) Any line of a plane separates the plane into two half-planes.
(3) Any point of a line separates the line into two half-lines.

There is one other definition that is useful. A ray is a half-line together with its endpoint. A ray has one endpoint. A ray without its endpoint is a half-line. We usually draw a ray like this, ⟶ . If A is the endpoint of a ray and B is another point of the ray, we denote the ray by $\overrightarrow{AB}$. Note that $\overrightarrow{BA}$ is not $\overrightarrow{AB}$. We use the term ray in the same sense in which it is used in "ray of light."

In everyday language, we sometimes do not distinguish between lines, rays, and segments. In geometry we should distinguish between them. A "line of sight" really refers to a ray. You do not describe somebody as in your line of sight if he is behind you. The right field foul line in baseball really refers to a segment and a ray. The segment extends from home plate through first base to the ball park fence. It stops at the fence. The ray starts on the ground and goes up the fence. What happens to a home run ball after it leaves the park makes no difference to the play in a major league game.

Exercises 7

1. In the figure at the right, is the R-side of ℓ the same as the S-side of ℓ? Is it the same as the Q-side? Are the intersections of ℓ and $\overline{PQ}$, ℓ and $\overline{RS}$ empty? Are the intersections of ℓ and $\overline{QS}$, ℓ and $\overline{PR}$ empty? Considering the sides of ℓ, are the previous two answers what you would expect?



2. Draw a line containing points A and B. What is $\overrightarrow{AB} \cap \overrightarrow{BA}$? What is the set of points not in $\overrightarrow{AB}$?

3. Draw a horizontal line. Label four points of it A, B, C and D in that order from left to right.
   Name two rays (using these points for their description):
   (a) Whose union is the line.
   (b) Whose union is not the line, but contains A, B, C, and D.
   (c) Whose union does not contain A.
   (d) Whose intersection is a point.
   (e) Whose intersection is empty.

4. Does a segment separate a plane? Does a line separate space?

5. Draw two horizontal lines k and ℓ on your paper. These lines are parallel. Label point P on ℓ. Is every point of ℓ on the P-side of k? Is this question the same as "Does the P-side of k contain ℓ"?

6. The idea of a plane separating space is related to the idea of the surface of a box separating the inside from the outside. If P is a point on the inside and Q a point on the outside of a box, does $\overline{PQ}$ intersect the surface?

*7. Explain how the union of two half-planes might be a plane.

*8. If A and B are points on the same side of the plane M (in space), must $\overleftrightarrow{AB} \cap M$ be empty? Can $\overleftrightarrow{AB} \cap M$ be empty?

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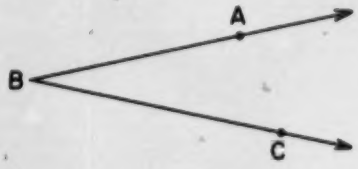
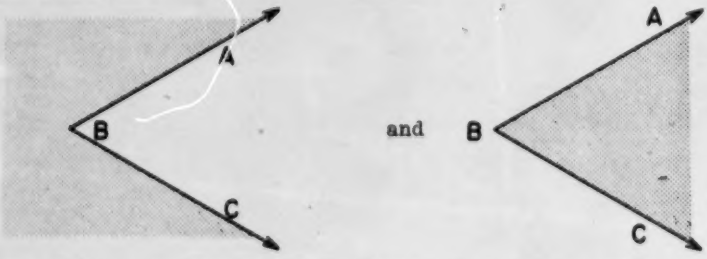
8. Angles and Triangles

Angles. Some of the most important ideas of geometry deal with angles and triangles. An angle is a set of points consisting of two rays with a common endpoint and not both on the same straight line. Let us say this in another way. Let $\overrightarrow{BA}$ and $\overrightarrow{BC}$ be two rays such that A, B, and C are not all on the same line. Then the set of points consisting of all the points of $\overrightarrow{BA}$ together with all the points of $\overrightarrow{BC}$ is called the angle ABC. The angle is the union of two rays. The point B is called the vertex of the angle. The rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are called the rays (or sometimes the sides) of the angle. An angle has exactly one vertex and exactly two rays.

An angle is drawn in the figure below. You will recall from section 3 that we really mean "a representation of an angle is drawn." Three points of the angle are labeled so that the angle is read "angle ABC" and may be written as, "$\angle$ABC." The letter of the vertex is always listed in the middle. Therefore, $\angle$ABC is $\angle$CBA. Note that in labeling this angle the order of A and C does not matter, but B must be in the middle. Is $\angle$ABC the same as $\angle$BAC (not drawn)?

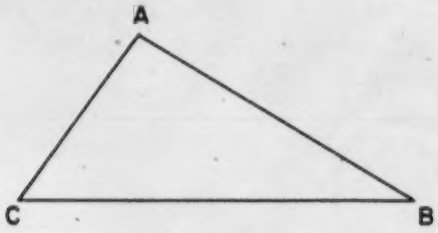


From the figure it looks as if the angle ABC separates the plane containing it. It is true that the angle does separate the plane. The two pieces into which the angle separates the plane look somewhat different. They look like:



We call the piece indicated on the right the interior of the angle and the one on the left the exterior. We can define the interior of the $\angle$ABC as the intersection of the A-side of the line $\overleftrightarrow{BC}$ and the C-side of the line $\overleftrightarrow{AB}$. It is the intersection of two half planes and does not include the angle. The exterior is the set of all points of the plane not on the angle or in the interior.

Triangles. Let A, B, and C be three points not all on the same straight line. The triangle ABC, written as $\triangle ABC$, is the union of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$. You will recall that the union of two sets consists of all of the elements of the one set together with all the elements of the other. We may define the $\triangle ABC$ in another way. The triangle ABC is the set of points consisting of A, B, and C, and all points of $\overleftrightarrow{AB}$ between A and B, all points of $\overleftrightarrow{AC}$ between A and C, and all points of $\overleftrightarrow{BC}$ between B and C. The points A, B, and C are the vertices of $\triangle ABC$. We say "vertices" when referring to more than one vertex. Triangle ABC is represented in the figure.



Angles of a Triangle. We speak of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ as the sides of the triangle. We speak of $\angle ABC$, $\angle ACB$, and $\angle BAC$ as the angles of the triangle. These are the angles determined by the triangle. Are the sides of the triangle contained in the triangle? Are the angles of a triangle contained in the triangle? You may wonder why we call $\angle ABC$ an angle of $\triangle ABC$ when $\angle ABC$ is not contained in $\triangle ABC$. The sides of the triangle are line segments; the sides of the angles are rays. The difference is important.

Note that a triangle is a set of points in exactly one plane. Every point of the triangle ABC is in the plane ABC. Look at the figure above. Does $\triangle ABC$ separate the plane in which it lies? Yes, it certainly seems to do so. It is true that it does. The $\triangle ABC$ has an interior and an exterior. The interior is the intersection of the interiors of the three angles of the triangle. The exterior is the set of all points of plane ABC not on $\triangle ABC$ or in the interior of $\triangle ABC$.

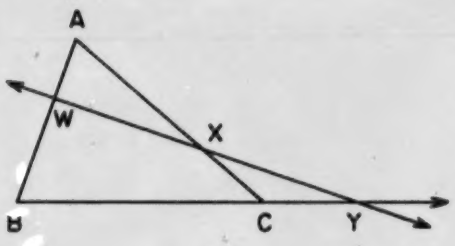
Exercises 8

1. Label three points A, B, and C not all on the same line. Draw $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$.
   (a) Shade the C-side of $\overleftrightarrow{AB}$. Shade the A-side of $\overleftrightarrow{BC}$. What set is now doubly shaded?
   (b) Shade the B-side of $\overleftrightarrow{AC}$. What set is now triply shaded?

2. Label three points X, Y, and Z not all on the same line.
   (a) Draw $\angle XYZ$ and $\angle XZY$. Are they different angles? Why?
   (b) Draw $\angle YXZ$. Is this angle different from both of the other two you have drawn?
   (c) Each angle is a set of points in exactly one plane. Why is this true?

3. Draw a triangle ABC.
   (a) In the triangle, what is $\overline{AB} \cap \overline{AC}$?
   (b) Does the triangle contain any rays or half-lines?
   (c) In the drawing extend $\overline{AB}$ in both directions to obtain $\overleftrightarrow{AB}$. What is $\overline{AB} \cap \overleftrightarrow{AB}$?
   (d) What is $\overleftrightarrow{AB} \cap \triangle ABC$?

4. Refer to the figure on the right.
   (a) What is $\overline{YW} \cap \triangle ABC$?
   (b) Name the four triangles in the figure.
   (c) Which of the labeled points, if any, are in the interior of any of the triangles?
   (d) Which of the labeled points, if any, are in the exterior of any of the triangles?
   (e) Name a point on the same side of $\overleftrightarrow{WY}$ as C and one on the opposite side.

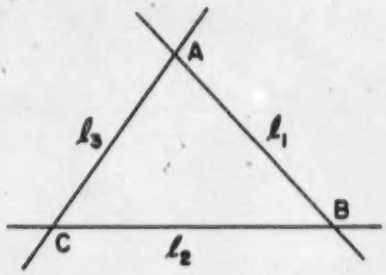


5. Draw a figure like that of Exercise 4.
   (a) Label a point P not in the interior of any of the triangles.
   (b) Label a point Q inside two of the triangles.
   (c) If possible, label a point R in the interior of $\triangle ABC$ but not in the interior of any other of the triangles.

6. If possible, make sketches in which the intersection of a line and a triangle is:
   (a) The empty set.
   (b) A set of two elements.
   (c) A set of one element.
   (d) A set of exactly three elements.

7. If possible, make sketches in which the intersection of two triangles is:
   (a) The empty set.
   (b) Exactly two points.
   (c) Exactly four points.
   (d) Exactly five points.

8. In the figure, what are the following:



(a) $\angle ABC \cap \overleftrightarrow{AC}$

(b) $\triangle ABC \cap \overleftrightarrow{AB}$

(c) $\ell_1 \cap \angle ACB$

(d) $\overline{AB} \cap \ell_2$

(e) $\angle BCA \cap \angle ACB$

(f) $\overline{BC} \cap \angle ABC$

(g) $\overrightarrow{BC} \cap \angle ACB$

(h) $\angle ABC \cap \triangle ABC$

*9. In a plane if two triangles have a side of each in common, must their interiors intersect? If three triangles have a side of each in common, must some two of their interiors intersect?

*10. Draw $\angle ABC$. Label points X and Y in the interior and P and Q in the exterior.

(a) Must every point of $\overline{XY}$ be in the interior?

(b) Is every point of $\overline{PQ}$ in the exterior?

(c) Can you find points R and S in the exterior so that $\overline{RS} \cap \angle ABC$ is not empty?

(d) Can $\overline{XP} \cap \angle ABC$ be empty?

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9. <u>One-to-One Correspondences</u>

In your previous mathematical experiences, you may have used the idea of "one-to-one correspondence" in talking about counting numbers. This idea is also useful in geometry. By "one-to-one correspondence" we mean the matching of each member of a certain set with a corresponding member of another set. Before we use this idea in geometry, let us review our previous experience.

Once upon a time, primitive man, as he kept track of his sheep, matched each sheep in his flock with a stone. The shepherd would build a pile of stones each morning, composed of one stone for each sheep, as the flock left the fold for the pasture. In the evening he would transfer the stones to a new pile, one at a time, as each sheep entered the fold. If all stones were thus transferred, then the shepherd knew that all his sheep were safe in the fold for the night. This is true because for each stone there was a matching sheep, and for each sheep there was a matching stone.

Let us take another example. Suppose you have sold seventeen tickets to a school play. Let T be the set of tickets you have sold. Let C be the set of people who are admitted to the theatre by these tickets. Is there a one-to-one correspondence between T and C? How do you know?

Consider the set of counting numbers less than eleven. Let us form two sets from these numbers. Set A, containing the odd numbers: {1,3,5,7,9} and set B, containing the even numbers: {2,4,6,8,10}. Is there a one-to-one correspondence between set A and set B? The answer, of course, is yes, because every odd number can be matched with an even number. Let us demonstrate this by use of the following scheme:

1	3	5	7	9
↕	↕	↕	↕	↕
2	4	6	8	10

Is this the only way in which the elements of these two sets can be matched? Form a different matching of your own.

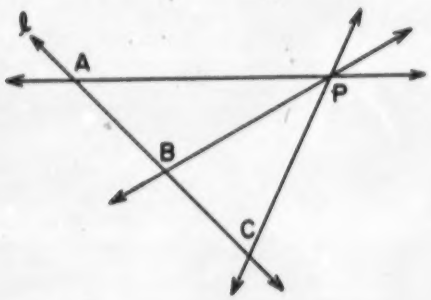
The game of musical chairs is fun because it is based on <u>not</u> having a one-to-one correspondence. Why is this? Is there ever a one-to-one correspondence at some stage in the game?

Given set A: {a,b,c,d,e,f,g}, and set B: {I,II,III,IV,V}, is there a one-to-one correspondence between these two sets? Demonstrate your answer by using the scheme used previously for the sets of even and odd numbers.

The facts you learn about number and the facts you learn in the study of geometry will eventually be combined in a subject known as "coordinate geometry." Great advances in science and technology have been made because of this combination. To understand how this union of number and geometry is accomplished, we shall once again call upon the ideas of "line in the geometric sense," "number line," and "one-to-one correspondence."

Class Discussion Problems

Follow the given directions in making a drawing somewhat like the given figure.



Questions are scattered throughout the directions. Answer the questions as you go along.

1. Draw a line and label it ℓ.

2. Choose a point not on line ℓ and label it P.

3. Mark some point A on line ℓ.

4. Draw line $\overleftrightarrow{PA}$.
   (a) Is $\overleftrightarrow{PA} \cap \ell$ equal to the empty set?
   (b) Does the intersection set of PA and ℓ have only one element? Why?

5. Choose two other points, B and C on ℓ. Draw $\overleftrightarrow{PB}$ and $\overleftrightarrow{PC}$.
   (a) Through each additional point marked on ℓ can you draw a line that also goes through point P?
   (b) Let all lines which intersect ℓ and pass through P be the elements of a set called K. How many elements of K have been drawn up to now?
   (c) Does each indicated element of K contain a point on ℓ?
   (d) Can each indicated element of K be matched with an indicated point on ℓ?
   (e) Do you think that, if more elements of K were drawn and more points on ℓ were marked, each element of K could be matched with a corresponding element of ℓ?

6. Mark points D and E on ℓ. Draw elements of K that can be matched with these points. Is it true that for each element (indicated or not) of K there is a corresponding element of ℓ and for each element of ℓ there is a corresponding element of K?

7. Draw an element of K.
   (a) Does it intersect ℓ?
   (b) In how many points does it intersect ℓ?

8. Copy and complete the following sentence to make it true:
   To a _____ through P and intersecting ℓ there corresponds a _____ on ℓ and to a _____ on ℓ there corresponds a _____ through P and intersecting ℓ. In other words, there is a one-to-one correspondence between the set K of lines and the set ℓ of points.

Exercises 9

1. Is there a one-to-one correspondence between the pupils and the desks in your room? Why?

2. Is there a one-to-one correspondence between the states of the United States and the U. S. cities of over 1,000,000 in population? Why?

3. You have heard of the expression: "Let's count noses." Does this imply a one-to-one correspondence situation? If so, what is it?

4. Show that there is a one-to-one correspondence between the set of even whole numbers and the set of odd whole numbers.

5. If set R is in one-to-one correspondence with set S and set S with set T, is set R in one-to-one correspondence with set T? Why?

6. Describe a one-to-one correspondence between the points A, B, and C which determine a triangle and the sides of the triangle. Can you do this in more than one way?

7. Draw a triangle with vertices A, B, and C. Label a point P in the interior of $\triangle ABC$. Let H be the set of all rays having P as endpoint. We understand that the elements of H are in the plane of $\triangle ABC$. Draw several rays of H. Can you observe a one-to-one correspondence between H and $\triangle ABC$? For every point of $\triangle ABC$ is there exactly one ray of H containing it? For every ray of H is there exactly one point of $\triangle ABC$ on such ray?

8. Draw an angle XYZ with the vertex at Y. Draw the segment $\overline{XZ}$. Think of K as a set of rays in plane XYZ with common endpoint at Y. K is the set of all such rays which do <u>not</u> contain points in the exterior of $\angle XYZ$. $\overrightarrow{YX}$ and $\overrightarrow{YZ}$ are two of the many elements of K. Draw another element of K. Does it intersect $\overline{XZ}$? For each element of K will there be one such matching point of $\overline{XZ}$? Label D a point of $\overrightarrow{YX}$ and E a point of $\overrightarrow{YZ}$. Draw $\overline{DE}$. Is there a similar one-to-one correspondence between the set of points of $\overline{YZ}$ and the set of points of $\overline{DE}$?

*9. Describe a one-to-one correspondence between the set of points on the surface of a ball and the set of rays with common endpoint inside the ball.

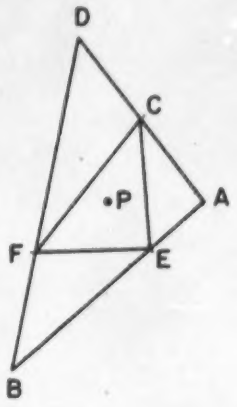
*10. Describe a one-to-one correspondence between a set H of all lines in a plane through a point and a set K of all planes through a line in space. (Think of a revolving door and the floor under the door.)

*11. Let S be the set of all rays in plane ABD with endpoint at P.

(a) Is there a one-to-one correspondence between S and $\triangle ABD$?

(b) Is there a one-to-one correspondence between S and $\triangle FCE$?

(c) Is there a one-to-one correspondence between $\triangle ABD$ and $\triangle FCE$? Why?



*12. Establish a one-to-one correspondence between the set of all even numbers and the set of all natural numbers.

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10. Simple Closed Curves

In newspapers and magazines you often see graphs like those in Figures a and b. These graphs represent what are called curves. We shall consider curves to be special types of sets of points. Sometimes paths that wander around in space are thought of as curves. In this section, however, we confine our attention to curves that are contained in one plane. We can represent them by figures we draw on a chalkboard or on a sheet of paper.

A curve is a set of points which can be represented by a pencil drawing made without lifting the pencil off the paper. Segments and triangles are examples of curves we have already studied. Curves may or may not contain portions that are straight. In everyday language we use the term "curve" in this same sense. When a baseball pitcher throws a curve, the ball seems to go straight for a while and then "breaks" or "curves."

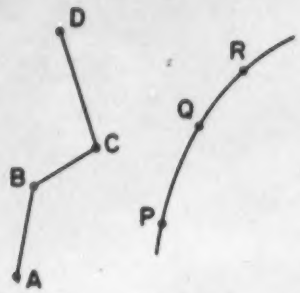


Fig. a

Fig. b

Fig. c

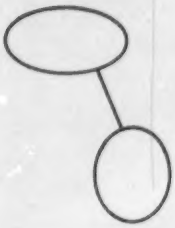


Fig. d

Fig. e

One important type of curve is called a broken-line curve. It is the union of several line segments; that is, it consists of all the points on several line segments. Figure a represents a broken-line curve. A, B, C, and D are indicated as points on the curve. We also say that the curve contains or passes through these points. Figures b to i also represent curves. In Figure b, points P, Q, and R are indicated on the curve. Of course, we think of the curve as containing many points other than P, Q, and R.

A curve is said to be a simple closed curve if it can be represented by a figure drawn in the following manner. The drawing starts and stops at the same point. Otherwise, no point is touched twice. Figures c, g, h, and i represent simple closed curves. The other figures of this section do not. Figure j represents two simple closed curves. The boundary of a state like Iowa or Utah on an ordinary map represents a simple closed curve. A fence which extends all the way around a cornfield suggests a simple closed curve.

Fig. f

Fig. g

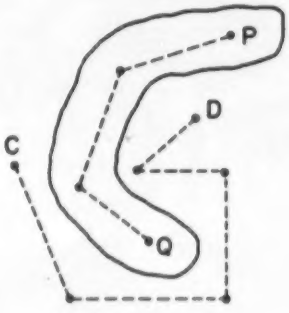


Fig. h

The examples we have mentioned, including that of a triangle, suggest a very important property of simple closed curves. Each simple closed curve has an interior and an exterior in the plane. Furthermore, any curve at all containing a point in the interior and a point in the exterior must intersect the simple closed curve. As an example, consider any curve containing

A and B of Figure g and lying in the plane. Also any two points in the interior (or any two points in the exterior) may be joined by a broken-line curve which does not intersect the simple closed curve. Figure h indicates this. A simple closed curve is the boundary of its interior and also of its exterior.

We call the interior of a simple closed curve a region. There are other types of sets in the plane which are also regions. In Figure j, the portion of the plane between the two simple closed curves is called a region. Usually a region (as a set of points) does not include its boundary.

Fig. i

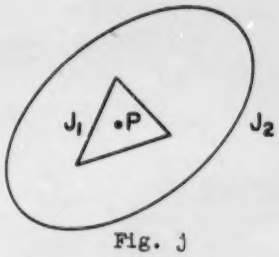


Fig. j

Consider Figure j. The simple closed curve (represented by) J_1 is in the interior of the simple closed curve J_2. We may obtain a natural one-to-one correspondence between J_1 and J_2 as follows. Consider a point such as P in the interior of J_1. Consider the set of rays with endpoint at P. Each such ray intersects each point of J_1 and J_2 in a single point. Furthermore, each point of J_1 and each point of J_2 is on one such ray. A point of J_1 corresponds to a point of J_2 if the two points are on the same ray from P. Note that this procedure using rays would not determine a one-to-one correspondence if one of the simple closed curves were like that in Figure i.

Exercises 10

1. Draw a figure representing two simple closed curves whose intersection is exactly two points. How many simple closed curves are represented in your figure?

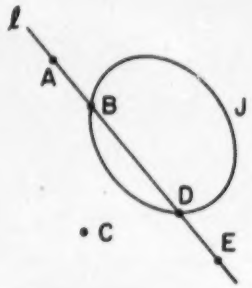
2. In Figure j, describe the region between the curves in terms of intersection, interior and exterior.

3. Draw two triangles whose intersection is a side of each. Is the union of the other sides of both triangles a simple closed curve? How many simple closed curves are represented in your figure?

4. In a map of the United States, does the union of the boundaries of Colorado and Arizona represent a simple closed curve?

5. Think of X and Y as bugs which can crawl anywhere in a plane. List three different simple sets of points in the plane, any one of which will provide a boundary which separates X and Y.

6. The line ℓ and the simple closed curve J are as shown in the figure.
   (a) What is $J \cap \ell$?
   (b) Draw a figure and shade the intersection of the interior of J and the C-side of ℓ.
   (c) Describe in terms of rays the set of points on ℓ <u>not</u> in the interior of J.



*7. Draw two simple closed curves, one in the interior of the other such that, for <u>no</u> point P do the rays from P establish a one-to-one correspondence between the two curves. Consider Figure 1.

*8. Draw two simple closed curves whose interiors intersect in three different regions.

*9. Explain why the intersection of a simple closed curve and a line cannot contain exactly three points if the curve crosses the line when it intersects it.